Polynilpotent Multipliers of Finitely Generated Abelian Groups *

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Abstract

In this paper, we present an explicit formula for the Baer invariant of a finitely generated abelian group with respect to the variety of polynilpotent groups of class row $(c_1, \ldots, c_t)$, $\mathcal{N}_{c_1,\ldots,c_t}$. In particular, one can obtain an explicit structure of the ℓ-solvable multiplier (the Baer invariant with respect to the vaiety of solvable groups of length at most $\ell \geq 1$, $\mathcal{S}_\ell$.) of a finitely generated abelian group.

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1. Introduction and preliminaries

I.Schur [13], in 1907, found a formula for the Schur multiplier of a direct product of two finite groups as follows:

$$M(A \times B) \cong M(A) \oplus M(B) \oplus A_{ab} \otimes B_{ab}.$$

One of the important corollaries of the above fact is an explicit formula for the Schur multiplier of a finite abelian group $G \cong \mathbf{Z}_{n_1} \oplus \mathbf{Z}_{n_2} \oplus \ldots, \mathbf{Z}_{n_k}$, where $n_{i+1} \mid n_i$ for all $1 \leq i \leq k-1$, as follows:

$$M(G) \cong \mathbf{Z}_{n_2} \oplus \mathbf{Z}_{n_3}^{(2)} \oplus \ldots \oplus \mathbf{Z}_{n_k}^{(k-1)},$$

where $\mathbf{Z}_n^{(m)}$ denotes the direct sum of m copies of the cyclic group $\mathbf{Z}_n$ (see [10]).

In 1997, the first author, in a joint paper [11], succeeded to generalize the above formula for the Baer invariant of a finite abelian group $G \cong \mathbf{Z}_{n_1} \oplus \mathbf{Z}_{n_2} \oplus \ldots, \mathbf{Z}_{n_k}$, where $n_{i+1} \mid n_i$ for all $1 \leq i \leq k-1$, with respect to the variety of nilpotent groups of class at most $c \geq 1$, $\mathcal{N}_c$, as follows:

$$\mathcal{N}_c M(G) \cong \mathbf{Z}_{n_2}^{(b_2)} \oplus \mathbf{Z}_{n_3}^{(b_3-b_2)} \oplus \ldots \oplus \mathbf{Z}_{n_k}^{(b_k-b_{k-1})},$$

where b_i is the number of basic commutators of weight $c+1$ on i letters (see [4]).

$\mathcal{N}_c M(G)$ is also called the c-nilpotent multiplier of G (see [3]). Note that, by a similar method of the paper [11], we can obtain the structure of the c-nilpotent multiplier of a finitely generated abelian group as the following theorem.

Theorem 1.1. Let $G \cong \mathbf{Z}^{(m)} \oplus \mathbf{Z}_{n_1} \oplus \mathbf{Z}_{n_2} \oplus \ldots \mathbf{Z}_{n_k}$ be a finitely generated abelian group, where $n_{i+1} \mid n_i$ for all $1 \leq i \leq k-1$, then

$$\mathcal{N}_c M(G) \cong \mathbf{Z}^{(b_m)} \oplus \mathbf{Z}_{n_1}^{(b_{m+1}-b_m)} \oplus \mathbf{Z}_{n_2}^{(b_{m+2}-b_{m+1})} \oplus \ldots \oplus \mathbf{Z}_{n_k}^{(b_{m+k}-b_{m+k-1})},$$

where b_i is the number of basic commutators of weight $c+1$ on i letters and $b_0 = b_1 = 0$.

Now, in this paper, we intend to generalize the above theorem to obtain an explicit formula for $\mathcal{N}_{c_1,\ldots,c_t}M(G)$, the Baer invariant of G with respect to the variety of polynilpotent groups of class row $(c_1, \ldots, c_t)$, $\mathcal{N}_{c_1,\ldots,c_t}$, where G is a finitely generated abelian group. We also call $\mathcal{N}_{c_1,\ldots,c_t}M(G)$, a polynilpotent multiplier of G. As an immediate consequence, one can obtain an explicit formula for the ℓ-solvable multiplier of G, $\mathcal{S}_\ell M(G)$.

Definition 1.2. Let G be any group with a free presentation $G \cong F/R$, where F is a free group. Then, after R. Baer [1], the Baer invariant of G with respect to a variety of groups $\mathcal{V}$, denoted by $\mathcal{V}M(G)$, is defined to be

$$\mathcal{V}M(G) = \frac{R \cap V(F)}{[RV^*F]} ,$$

where V is the set of words of the variety $\mathcal{V}$, $V(F)$ is the verbal subgroup of F with respect to $\mathcal{V}$ and

$$[RV^*F] =< v(f_1, \ldots, f_{i-1}, f_i r, f_{i+1}, \ldots, f_n)v(f_1, \ldots, f_i, \ldots, f_n)^{-1} \mid$$

$$r \in R, 1 \leq i \leq n, v \in V, f_i \in F, n \in \mathbf{N} > .$$

In special case, if $\mathcal{V}$ is the variety of abelian groups, $\mathcal{A}$, then the Baer invariant of G will be

$$\frac{R \cap F'}{[R, F]},$$

which, following Hopf [7], is isomorphic to the second cohomology group of G, $H_2(G, C^*)$, in finite case and also is isomorphic to the well-known notion the Schur multiplier of G, denoted by $M(G)$. The multiplier $M(G)$ arose in Schur's work [12] of 1904 on projective representation of a group, and has subsequently found a variety of other applications. The survey article of

Wiegold [14] and the books of Beyl and Tappe [2] and Karpilovsky [10] form a fairly comprehensive account of $M(G)$.

If $\mathcal{V}$ is the variety of nilpotent groups of class at most $c \geq 1$, $\mathcal{N}_c$, then the Baer invariant of G with respect to $\mathcal{N}_c$ will be

$$\mathcal{N}_c M(G) = \frac{R \cap \gamma_{c+1}(F)}{[R,\ _cF]},$$

where $\gamma_{c+1}(F)$ is the $(c+1)$-st term of the lower central series of F and $[R,\ _1F] = [R, F], [R,\ _cF] = [[R,\ _{c-1}F], F]$, inductively.

If $\mathcal{V}$ is the variety of solvable groups of length at most $\ell \geq 1$, $\mathcal{S}_\ell$, then the Baer invariant of G with respect to $\mathcal{S}_\ell$ will be

$$\mathcal{S}_\ell M(G) = \frac{R \cap \delta^l(F)}{[R, F, \delta^1(F), \ldots, \delta^{l-1}(F)]},$$

where $\delta^i(F)$ is the i-th derived subgroup of F.See [8,corollary 2.10] for the equality $[RS_l^*F] = [R, F, \delta^1(F), \ldots, \delta^{l-1}(F)]$.

In a very more general case, let $\mathcal{V}$ be the variety of polynilpotent groups of class row $(c_1, \ldots, c_t)$, $\mathcal{N}_{c_1,\ldots,c_t}$, then the Baer invariant of a group G with respect to this variety is as follows:

$$\mathcal{N}_{c_1,\ldots,c_t} M(G) \cong \frac{R \cap \gamma_{c_t+1} \circ \ldots \circ \gamma_{c_1+1}(F)}{[R,\ _{c_1}F,\ _{c_2}\gamma_{c_1+1}(F), \ldots,\ _{c_t}\gamma_{c_{t-1}+1} \circ \ldots \circ \gamma_{c_1+1}(F)]}\ ,$$

where $\gamma_{c_t+1} \circ \ldots \circ \gamma_{c_1+1}(F) = \gamma_{c_t+1}(\gamma_{c_{t-1}+1}(\ldots(\gamma_{c_1+1}(F))\ldots))$ are the terms of iterated lower central series of F. See [6, Corollary 6.14] for the equality

$$[R\mathcal{N}^*_{c_1,\ldots,c_t}F] = [R,\ _{c_1}F,\ _{c_2}\gamma_{c_1+1}(F), \ldots,\ _{c_t}\gamma_{c_{t-1}+1} \circ \ldots \circ \gamma_{c_1+1}(F)].$$

In the following, we are going to mention some definitions and notations of T.C. Hurley and M.A. Ward [9], which are vital in our investigation.

Definition and Notation 1.3. Commutators are written $[a, b] = a^{-1}b^{-1}ab$ and the usual convention for left-normed commutators is used, $[a, b, c] = [[a, b], c],\ [a, b, c, d] = [[[a, b], c], d]$ and so on, including the trivial case $[a] = a$.

Basic commutators are defined in the usual way. If X is a fully ordered independent subset of a free group, the basic commutators on X are defined inductively over their weight as follows:
(i) All the members of X are basic commutators on X of weight one on X.
(ii) Assuming that $n > 1$ and that the basic commutators of weight less than n on X have been defined and ordered.
(iii) A commutator $[a, b]$ is a basic commutator of weight n on X if $wt(a) + wt(b) = n,\ a < b$, and if $b = [b_1, b_2]$, then $b_2 \leq a$. The ordering of basic commutators is then extended to include those of weight n in any way such that those of weight less than n precede those of weight n. The natural way to define the order on basic commutators of the same weight is lexicographically, $[b_1, a_1] < [b_2, a_2]$ if $b_1 < b_2$ or if $b_1 = b_2$ and $a_1 < a_2$.

A word of the form

$$[c, a_1, a_2, \ldots, a_p, b_1^{\beta_1}, b_2^{\beta_2}, \ldots, b_q^{\beta_q}]$$

is a *"standard invertator"* will be meant to imply that the β_i's are ± 1, $c > a_1 \leq a_2 \leq \ldots \leq a_p \leq b_1 \leq b_2 \leq \ldots \leq b_q$ and if $b_i = b_j$ then $\beta_i = \beta_j$ for all i, j. Whenever this terminology is used it will be accomplished by a statement of what set X, the a_i and the b_j are chosen from and this will be always be a set which is known to be fully ordered in some way. Restrictions on the values of p and q will be given, the value $p = 0$ and $q = 0$ being permissible so that we may, when we wish, specify standard invertators of the forms $[c, a_1, \ldots, a_p]$ or $[c, b_1^{\beta_1}, b_2^{\beta_2}, \ldots, b_q^{\beta_q}]$.

Let F be a free group on alphabet X and m and n be integers. Then
(i) $A_{m,n}$ denotes the set of all basic commutators on X of weight exactly n and of the form $[c, a_1, \ldots, a_p]$, where b and the a_i are all basic commutators

on X of weight less than m.
(ii) $B_{m,n}$ denotes the set of all standard invertators on X of the form

$$[b, a_1, a_2, \dots, a_p, a_{p+1}^{\alpha_{p+1}}, \dots, a_q^{\alpha_q}],$$

where $0 \le p < q$, b and the a_i are basic commutators on X of weight less than m,

$$wt([b, a_1, a_2, \dots, a_p]) < n \le wt([b, a_1, a_2, \dots, a_p, a_{p+1}^{\alpha_{p+1}}])$$

and $b = [b_1, b_2]$ implies $b_2 \le a_1$. Note that $[b, a_1, a_2, \dots, a_p] \in A_{m,r}$, where r is the weight of this commutator and $r < n$. Also, observe that $A_{m,m}$ is just the set of all basic commutators of weight m on X.

Theorem 1.4 (P.Hall [4,5]). Let $F =< x_1, x_2, \dots, x_d >$ be a free group, then

$$\frac{\gamma_n(F)}{\gamma_{n+i}(F)} \quad , \qquad 1 \le i \le n$$

is the free abelian group freely generated by the basic commutators of weights $n, n+1, \dots, n+i-1$ on the letters $\{x_1, \dots, x_d\}$.

Theorem 1.5 (Witt Formula [4]). The number of basic commutators of weight n on d generators is given by the following formula:

$$\chi_n(d) = \frac{1}{n} \sum_{m|n} \mu(m) d^{n/m}$$

where $\mu(m)$ is the *Mobious function*, and defined to be

$$\mu(m) = \begin{cases} 1 & \text{if } m = 1, \\ 0 & \text{if } m = p_1^{\alpha_1} \dots p_k^{\alpha_k} \quad \exists \alpha_i > 1, \\ (-1)^s & \text{if } m = p_1 \dots p_s, \end{cases}$$

The following important theorem presents interesting free generating sets for the terms of the lower central series of a free group which will be used several times in this paper.

Theorem 1.6 (T.C.Hurley and M.A.Ward 1981). Let F be a free group, freely generated by some fully ordered set X, and let m and n be integers satisfying $2 \leq m \leq n$. Then the members of the set

$$A_{m,m} \cup A_{m,m+1} \cup \ldots \cup A_{m,n-1} \cup B_{m,n}$$

are distinct as written, so that in particular this is a disjoint union, and the set freely generates $\gamma_m(F)$.

Proof. See [9, Theorem 2.2].

Corollary 1.7. Let F be a free group freely generated by some fully ordered set X. Then $\gamma_{c_2+1}(\gamma_{c_1+1}(F))$ is freely generated by

$$\hat{A}_{c_2+1,c_2+1} \cup \hat{B}_{c_2+1,c_2+2},$$

where $\hat{A}_{c_2+1,c_2+1}$ is the set of all basic commutators of weight c_2+1 on the set

$$Y = A_{c_1+1,c_1+1} \cup B_{c_1+1,c_1+2},$$

and $\hat{B}_{c_2+1,c_2+2}$ is the set of all standard invertators on Y of the form

$$[b, a_1, a_2, \ldots, a_p, a_{p+1}^{\alpha_{p+1}}, \ldots, a_q^{\alpha_q}],$$

where $0 \leq p < q$, b and the a_i are basic commutators on Y of weight less than c_2+1,

$$wt([b, a_1, a_2, \ldots, a_p]) < c_2 + 2 \leq wt([b, a_1, a_2, \ldots, a_p, a_{p+1}^{\alpha_{p+1}}])$$

and $b=[b_1,b_2]$ implies $b_2 \le a_1$.

Proof. Using Theorem 1.6, $\gamma_{c_1+1}(F)$ is freely generated by $A_{c_1+1,c_1+1} \cup B_{c_1+1,c_1+2}$, when putting $m=c_1+1, n=c_1+2$. Now we can suppose $\overline{F}=\gamma_{c_1+1}(F)$ is a free group, freely generated by fully ordered set $Y=A_{c_1+1,c_1+1} \cup B_{c_1+1,c_1+2}$. Applying Theorem 1.6 again for $\gamma_{c_2+1}(\overline{F})$ and $m=c_2+1, n=c_2+2$, the result holds. $\Box$

As an immediate consequence we have the following corollary.

Corollary 1.8. Let F be a free group freely generated by some fully ordered set X. Then the second derived subgroup of F, $\delta^2(F)=F''$, is freely generated by

$$\hat{A}_{2,2} \cup \hat{B}_{2,3},$$

where $\hat{A}_{2,2}$ is the set of all basic commutators of weight 2 on the set $A_{2,2} \cup B_{2,3}$, and $\hat{B}_{2,3}$ is the set of all standard invertators on $A_{2,2} \cup B_{2,3}$ of the form $[b, a_1, a_2^{\alpha_2}, \ldots, a_q^{\alpha_q}]$, where $b, a_i \in A_{2,2} \cup B_{2,3}$.

2. The Main Results

In this section, first, we concentrate on the calculation of the Baer invariant of a finitely generated abelian group with respect to the variety of metabelian groups, *i.e.* solvable groups of length 2, $\mathcal{S}_2$.

Let $\mathbf{Z}_{r_i} =< x_i \mid x_i^{r_i} >, 1 \le i \le t$, be cyclic groups of order $r_i \ge 0$, and let

$$0 \longrightarrow R_i =< x_i^{r_i} > \longrightarrow F_i =< x_i > \longrightarrow \mathbf{Z}_{r_i} \longrightarrow 0\ ,$$

be a free presentation of $\mathbf{Z}_{r_i}$. Also, suppose $G \cong \oplus\sum_{i=1}^{t} \mathbf{Z}_{r_i}$ is the direct sum of the cyclic groups $\mathbf{Z}_{r_i}$. Then

$$0 \longrightarrow R \longrightarrow F \longrightarrow G \longrightarrow 0$$

is a free presentation of G, where $F = \prod_{i=1}^{*t} F_i =< x_1, \ldots, x_t >$ is the free product of F_i's, and $R = \prod_{i=1}^{t} R_i \gamma_2(F)$. Therefore, the metabelian multiplier of G is as follows:

$$\mathcal{S}_2 M(G) \cong \frac{R \cap \delta^2(F)}{[R, F, \delta^1(F)]} = \frac{F''}{[R, F, F']} \qquad (since \quad F' \leq R).$$

Now, the following theorem presents an explicit structure for the metabelian multiplier of a finitely generated abelian group.

Theorem 2.1. With the above notation and assumption, let $G \cong \mathbf{Z}^{(m)} \oplus \mathbf{Z}_{n_1} \oplus \ldots \oplus \mathbf{Z}_{n_k}$ be a finitely generated abelian group, where $n_{i+1} \mid n_i$ for all $1 \leq i \leq k-1$. Then the following isomorphism holds:

$$\mathcal{S}_2 M(G) \cong \mathbf{Z}^{(d_m)} \oplus \mathbf{Z}_{n_1}^{(d_{m+1}-d_m)} \oplus \ldots \oplus \mathbf{Z}_{n_k}^{(d_{m+k}-d_{m+k-1})},$$

where $d_i = \chi_2(\chi_2(i))$, and $\chi_2(i)$ is the number of basic commutators of weight 2 on i letters.

Proof. With the previous notation, put $t = m + k$, $r_1 = r_2 = \ldots = r_m = 0$, $r_{m+j} = n_j$, $1 \leq j \leq k$. Then $\mathbf{Z}_{r_1} \cong \ldots \cong \mathbf{Z}_{r_m} \cong \mathbf{Z}$, $\mathbf{Z}_{r_{m+j}} \cong \mathbf{Z}_{n_j}$, $G \cong \oplus \sum_{i=1}^{m+k} \mathbf{Z}_{r_i}$, and

$$\mathcal{S}_2 M(G) \cong \frac{F''}{[R, F, F']},$$

where F is the free group on the set $X = \{x_1, \ldots, x_m, x_{m+1}, \ldots, x_{m+k}\}$.

By corollary 1.8 F'' is a free group with the basis $\hat{A}_{2,2} \cup \hat{B}_{2,3}$. Put L the normal closure of those elements of the basis F'', $\hat{A}_{2,2} \cup \hat{B}_{2,3}$, of weight, as commutators on the set $X = \{x_1, \ldots, x_m, x_{m+1}, \ldots, x_{m+k}\}$, greater than 4 in F''. In other words

$$L =< w \in \hat{A}_{2,2} \cup \hat{B}_{2,3} \mid w \notin \{u \in \hat{A}_{2,2} \cup \hat{B}_{2,3} \mid u \ is \ of \ the \ form \ [[x_{i_1}, x_{i_2}], [x_{i_3}, x_{i_4}]]\} >^{F''} .$$

It is easy to see that F''/L is a free group freely generated by the following set

$$Y = \{wL \mid w \in \hat{A}_{2,2} \cup \hat{B}_{2,3} \ \ and\ w \ \ is \ \ of \ \ the \ \ form \ \ [[x_{i_1}, x_{i_2}], [x_{i_3}, x_{i_4}]]\}.$$

Therefore

$$\frac{F''/L}{(F''/L)'} \cong \frac{F''}{LF'''}$$

is a free abelian group with the basis $\overline{Y} = \{wLF''' \mid wL \in Y\}$. Since $\mathcal{S}_2M(G) \cong F''/[R, F, F']$ is abelian, so $F''' \leq [R, F, F']$. Thus, we have

$$\mathcal{S}_2M(G) \cong \frac{F''/LF'''}{[R, F, F']/LF'''}.$$

Now we are going to describe explicitly the bases of the free abelian group F''/LF''' and its subgroup $[R, F, F']/LF'''$ in order to find the structure of the metabelian multiplier of G, $\mathcal{S}_2M(G)$. According to the basis $\overline{Y}$ of the free abelian group F''/LF''', it is easy to see that

$$\overline{Y} = C_0 \cup C_1 \cup \ldots \cup C_k \ ,$$

where

$$C_0 = \{wLF''' \in \overline{Y} \mid w = [[x_{i_1}, x_{i_2}], [x_{i_3}, x_{i_4}]], 1 \leq i_1, i_2, i_3, i_4 \leq m\},$$

and for all $1 \leq \lambda \leq k$

$$C_\lambda = \{wLF''' \in \overline{Y} \mid w = [[x_{i_1}, x_{i_2}], [x_{i_3}, x_{i_4}]], 1 \leq i_1, i_2, i_3, i_4 \leq m + \lambda$$

$$, \exists\ 1 \leq j \leq 4, \ \ s.t.\ i_j = m + \lambda\}.$$

In order to find an appropriate basis for the free abelian group $[R, F, F']/LF'''$, first we claim that $\gamma_5(F) \cap F'' \leq LF'''$ (*), since, let $u \in F''$, using the basis $\overline{Y}$ of the free abelian group F''/LF''', we have

$$uLF''' = w_{i_1}^{\epsilon_1} \ldots w_{i_t}^{\epsilon_t} LF''',$$

where $w_{i_1}LF''',\ldots,w_{i_t}LF''' \in \overline{Y}$, and $\epsilon_1,\ldots,\epsilon_t \in \mathbf{Z}$. Clearly $LF''' \leq \gamma_5(F)$, so, if $u \in \gamma_5(F)$, then we have $w_{i_1}^{\epsilon_1}\ldots w_{i_t}^{\epsilon_t} \in \gamma_5(F)$. It is easy to see that $w_{i_1}^{\epsilon_1},\ldots,w_{i_t}^{\epsilon_t}$ are basic commutators of weight 4 on X. By Theorem 1.4 $\gamma_4(F)/\gamma_5(F)$ is the free abelian group with basis of all basic commutators of weight 4 on X. Thus we have $\epsilon_1 = \ldots = \epsilon_t = 0$, and hence $u \in LF'''$. As an immediate consequence we have $[F',F,F'] \leq LF'''$. Note that $R = (\prod_{i=1}^{m+k} R_i)F'$, where $R_i =< x_i^0 >= 1$, for all $1 \leq i \leq m$, and $R_{m+j} =< x_{m+j}^{n_j} >$, for all $1 \leq j \leq k$, so

$$\frac{[R,F,F']}{LF'''} = \frac{\prod_{j=1}^{k}[R_{m+j},F,F']LF'''}{LF'''} \ .$$

Using the above equality and the congruence

$$[[x_{i_1}^{\alpha_1},x_{i_2}^{\alpha_2}],[x_{i_3}^{\alpha_3},x_{i_4}^{\alpha_4}]] \equiv [[x_{i_1},x_{i_2}],[x_{i_3},x_{i_4}]]^{\alpha_1\alpha_2\alpha_3\alpha_4} \quad (mod \quad LF''') \ ,$$

for all $\alpha_1,\alpha_2,\alpha_3,\alpha_4 \in \mathbf{Z}$, (By (*)), it is routine to check that the free abelian group $[R,F,F']/LF'''$ has the following basis

$$D_1 \cup D_2 \cup \ldots \cup D_k \ ,$$

where $D_\lambda = \{w^{n_\lambda}LF''' \mid wLF''' \in C_\lambda,\ 1 \leq \lambda \leq k\}$.

Using the form of the elements C_λ and the number of basic commutators of weight 2 on i letters, $\chi_2(i)$, one can easily see that $\mid C_0 \mid = \chi_2(\chi_2(m))$, and $\mid C_\lambda \mid = \chi_2(\chi_2(m+\lambda)) - \chi_2(\chi_2(m+\lambda-1))$. Hence the result holds.□

Now, trying to generalize the proof of the above theorem, which is the basic idea of the paper, we are going to present an explicit formula for the polynilpotent multiplier of a finitely generated abelian group with respect to the variety $\mathcal{N}_{c_1,\ldots,c_t}$. Because of applying an iterative method and avoiding complicacy for the reader, first, we state and prove the beginning step of the

method for the variety $\mathcal{N}_{c_1,c_2}$ in the following theorem.

Theorem 2.2. Let $\mathcal{N}_{c_1,c_2}$ be the polynilpotent variety of class row (c_1, c_2) and $G \cong \mathbf{Z}^{(m)} \oplus \mathbf{Z}_{n_1} \oplus \ldots \oplus \mathbf{Z}_{n_k}$ be a finitely generated abelian group, where $n_{i+1} \mid n_i$ for all $1 \leq \imath \leq k-1$. Then the following isomorphism holds:

$$\mathcal{N}_{c_1,c_2}M(G) \cong \mathbf{Z}^{(e_m)} \oplus \mathbf{Z}_{n_1}^{(e_{m+1}-e_m)} \oplus \ldots \oplus \mathbf{Z}_{n_k}^{(e_{m+k}-e_{m+k-1})},$$

where $e_i = \chi_{c_2+1}(\chi_{c_1+1}(i))$ for all $m \leq i \leq m+k$.

Proof. By the notation of the Theorem 2.1 we have

$$\mathcal{N}_{c_1,c_2}M(G) = \frac{\gamma_{c_2+1}(\gamma_{c_2+1}(F))}{[R,\ _{c_1}F,\ _{c_2}\gamma_{c_1+1}(F)]},$$

where F is the free group on the set $X = \{x_1, \ldots, x_m, x_{m+1}, \ldots, x_{m+k}\}$. By considering the basis of the free group $\gamma_{c_2+1}(\gamma_{c_2+1}(F))$ presented in corollary 1.7, we put

$$L =< w \in \hat{A}_{c_2+1,c_2+1} \cup \hat{B}_{c_2+1,c_2+2} \mid \ w \notin E >^{\gamma_{c_2+1}(\gamma_{c_1+1}(F))},$$

where E is the set of all basic commutators of weight exactly c_2+1 on the set of all basic commutators of weight exactly c_1+1 on the set X.

Clearly $\gamma_{c_2+1}(\gamma_{c_1+1}(F))/L$ is free on the set

$$Y = \{wL \mid \ w \in \hat{A}_{c_2+1,c_2+1} \cup \hat{B}_{c_2+1,c_2+2} \textit{ and } w \in E\}$$

and $\gamma_{c_2+1}(\gamma_{c_1+1}(F))/L\gamma_{c_2+1}(\gamma_{c_1+1}(F))$ is free abelian with the basis $\bar{Y} = \{wL\gamma_{c_2+1}(\gamma_{c_1+1}(F)) \mid \ wL \in Y\}$. Considering the form of the elements of L and noticing to the abelian group $\mathcal{N}_{c_1,c_2}M(G)$, we have

$$L\gamma_{c_2+1}(\gamma_{c_1+1}(F)) \leq [R,\ _{c_1}F,\ _{c_2}\gamma_{c_1+1}(F)].$$

Thus the following isomorphism holds:

$$\mathcal{N}_{c_1,c_2}M(G) \cong \frac{\gamma_{c_2+1}(\gamma_{c_1+1}(F))/L\gamma_2(\gamma_{c_2+1}(\gamma_{c_1+1}(F)))}{[R,\ _{c_1}F,\ _{c_2}\gamma_{c_1+1}(F)]/L\gamma_2(\gamma_{c_2+1}(\gamma_{c_1+1}(F)))}.$$

By Theorem 1.4 $\gamma_{c_1+c_2+c_1c_2+1}(F)/\gamma_{c_1+c_2+c_1c_2+2}(F)$ is the free abelian group with the basis of all basic commutators of weight $c_1+c_2+c_1c_2+1$ on X. Using the above fact and the basis $\bar{Y}$ of the free abelian group $\gamma_{c_2+1}(\gamma_{c_1+1}(F))/L\gamma_{c_2+1}(\gamma_{c_1+1}(F))$ we can conclude the following inclusion:

$$\gamma_{c_1+c_2+c_1c_2+2}(F) \cap \gamma_{c_2+1}(\gamma_{c_1+1}(F)) \leq L\gamma_2(\gamma_{c_2+1}(\gamma_{c_1+1}(F))).$$

Now, it is easy to see that $\bar{Y} = C_0 \cup C_1 \cup \ldots \cup C_k$ is a basis for the free abelian group $\gamma_{c_2+1}(\gamma_{c_1+1}(F))/L\gamma_2(\gamma_{c_2+1}(\gamma_{c_1+1}(F)))$ and $D_1 \cup D_2 \cup \ldots \cup D_k$ is a basis for the free abelian group

$$\frac{[R,\ _{c_1}F,\ _{c_2}\gamma_{c_1+1}(F)]}{L\gamma_2(\gamma_{c_2+1}(\gamma_{c_1+1}(F)))},$$

where

$$C_0 = \{wL\gamma_2(\gamma_{c_2+1}(\gamma_{c_1+1}(F))) \in \bar{Y} \mid\ w \in E$$
$$\text{and } w \text{ is a commutator on letters } x_1, \ldots, x_m\},$$

and for $1 \leq \lambda \leq k$:
$C_\lambda = \{wL\gamma_2(\gamma_{c_2+1}(\gamma_{c_1+1}(F))) \in \bar{Y} \mid\ w \in E$ and w is a commutator on letters $x_1, \ldots, x_m, x_{m+1}, \ldots, x_{m+\lambda}$ such that the letter $x_{m+\lambda}$ does appear in $w\}$,
$D_\lambda = \{w^{n_\lambda}L\gamma_2(\gamma_{c_2+1}(\gamma_{c_1+1}(F))) \in \bar{Y} \mid\ wL\gamma_2(\gamma_{c_2+1}(\gamma_{c_1+1}(F))) \in C_\lambda\}$.
Note that using the form of the elements of C_λ and the number of basic commutators, we can conclude that $\mid C_0 \mid = \chi_{c_2+1}(\chi_{c_1+1}(m))$ and $\mid C_\lambda \mid = \chi_{c_2+1}(\chi_{c_1+1}(m+\lambda)) - \chi_{c_2+1}(\chi_{c_1+1}(m+\lambda-1))$. Hence the result holds. □

Now, we are ready to state and prove the main result of the paper in general case.

Theorem 2.3. Let $\mathcal{N}_{c_1,c_2,\ldots,c_t}$ be the polynilpotent variety of class row $(c_1, c_2, \ldots, c_t)$ and $G \cong \mathbf{Z}^{(m)} \oplus \mathbf{Z}_{n_1} \oplus \ldots \oplus \mathbf{Z}_{n_k}$ be a finitely generated abelian group, where $n_{i+1} \mid n_i$ for all $1 \leq \imath \leq k-1$. Then an explicit structure of the polynilpotent multiplier of G is as follows.

$$\mathcal{N}_{c_1,c_2,\ldots,c_t}M(G) \cong \mathbf{Z}^{(f_m)} \oplus \mathbf{Z}_{n_1}^{(f_{m+1}-f_m)} \oplus \ldots \oplus \mathbf{Z}_{n_k}^{(f_{m+k}-f_{m+k-1})},$$

where $f_i = \chi_{c_t+1}(\chi_{c_{t-1}+1}(\ldots(\chi_{c_1+1}(i))\ldots))$ for all $m \leq i \leq m+k$.

Proof. Let F be the free group on the set $X = \{x_1, \ldots, x_m, x_{m+1}, \ldots, x_{m+k}\}$. Then by previous notation, we have

$$\mathcal{N}_{c_1,c_2,\ldots,c_t}M(G) = \frac{\gamma_{c_t+1}(\gamma_{c_{t-1}+1}(\ldots(\gamma_{c_1+1}(F))\ldots))}{[R,\ _{c_1}F,\ _{c_2}\gamma_{c_1+1}(F), \ldots,\ _{c_t}\gamma_{c_t+1}(\gamma_{c_{t-1}+1}(\ldots(\gamma_{c_1+1}(F))\ldots))]}.$$

We define $\rho_t(F), E_t, X_t$ inductively on t as follows:
$\rho_1(F) = \gamma_{c_1+1}(F),\ \rho_i(F) = \gamma_{c_i+1}(\rho_{i-1}(F);$
$E_1 = X,\ E_i =$the set of all basic commutators of weight $c_i + 1$ on the set E_{i-1};
$X_1 = A_{c_1+1,c_1+1} \cup B_{c_1+1,c_1+2},\ X_i = \hat{A}_{c_i+1,c_i+1} \cup \hat{B}_{c_i+1,c_i+2}$, where $\hat{A}_{c_i+1,c_i+1}$ is the set of all basic commutators of weight $c_i + 1$ on the set X_{i-1}, and $\hat{B}_{c_i+1,c_i+2}$ is the set of all standard invertators on X_{i_1} of the form

$$[b, a_1, \ldots, a_p, a_{p+1}^{\alpha_{p+1}}, \ldots, a_q^{\alpha_q}],$$

where $0 \leq p < q$, b and the a_i are basic commutators on X_{i-1} of weight less than $c_i + 1$, $wt([b, a_1, \ldots, a_p]) < c_i + 2 \leq wt([b, a_1, \ldots, a_p, a_{p+1}^{\alpha_{p+1}}])$ and $b = [b_1, b_2]$ implies $b_2 \leq a_1$.

Using Theorem 1.6 and induction on t, it is easy to see that $\gamma_{c_t+1}(\gamma_{c_{t-1}+1}(\ldots(\gamma_{c_1+1}(F))\ldots)) = \rho_t(F)$ is freely generated by X_t. Now, putting

$$L_t =< w \in X_t \mid\ w \notin E_t >^{\rho_t(F)},$$

one can easily see that $\rho_t(F)/L_t$ is free on the set $Y_t = \{wL_t \mid\ w \in E_t\}$ and $\rho_t(F)/L_t\gamma_2(\rho_t(F))$ is free abelian with the basis $\bar{Y}_t = \{wL_t\gamma_2(\rho_t(F)) \mid\ w \in E_t\}$. By considering the abelian group $\mathcal{N}_{c_1,c_2,\ldots,c_t}M(G)$ and the form of the elements of L_t, we have $L_t\gamma_2(\rho_t(F)) \leq [R,\ _{c_1}F,\ _{c_2}\rho_1(F),\ldots,\ _{c_t}\rho_{t-1}(F)]$, and the following isomorphism

$$\mathcal{N}_{c_1,c_2,\ldots,c_t}M(G) \cong \frac{\rho_t(F)/L_t\gamma_2(\rho_t(F))}{[R,\ _{c_1}F,\ _{c_2}\rho_1(F),\ldots,\ _{c_t}\rho_{t-1}(F)]/L_t\gamma_2(\rho_t(F))}.$$

Clearly $\gamma_\pi(F)/\gamma_{\pi+1}(F)$ is the free abelian group with the basis of all basic commutators of weight π on X, where $\pi = \prod_{i=1}^{t}(c_i+1)$. Using the above fact and $\bar{Y}_t$, the basis of the free abelian group $\rho_t(F)/L_t\gamma_2(\rho_t(F))$, one can obtain the following inclusion:

$$\gamma_{\pi+1}(F) \cap \rho_t(F) \leq L_t\gamma_2(\rho_t(F)).$$

Therefore, it is clear that $\bar{Y}_t = C_{0,t} \cup C_{1,t} \cup \ldots \cup C_{k,t}$ is a basis for the free abelian group $\rho_t(F)/L_t\gamma_2(\rho_t(F))$ and $D_{0,t} \cup D_{1,t} \cup \ldots \cup D_{k,t}$ is a basis for the free abelian group $[R,_{c_1} F,_{c_2} \rho_1(F),\ldots,_{c_t} \rho_{t-1}(F)]/L_t\gamma_2(\rho_t(F))$, where

$$C_{0,t} = \{wL_t\gamma_2(\rho_t(F)) \in \bar{Y}_t \mid w \in E_t \text{ and } w \text{ is a}$$
$$\text{commutator on letters } x_1,\ldots,x_m\};$$
$$C_{\lambda,t} = \{wL_t\gamma_2(\rho_t(F)) \in \bar{Y}_t \mid\ w \in E_t \text{ and } w \text{ is a commutator on letters}$$
$$x_1,\ldots,x_m,x_{m+1},\ldots,x_{m+\lambda} \text{ such that the letter } x_{m+\lambda} \text{ does appear in } w\};$$
$$D_{\lambda,t} = \{w^{n_\lambda}L_t\gamma_2(\rho_t(F)) \mid\ wL_t\gamma_2(\rho_2(F)) \in C_{\lambda,t}\};$$
$$\text{for all } 1 \leq \lambda \leq k.$$

Note that $\mid C_{0,t} \mid= \chi_{c_t+1}(\ldots(\chi_{c_1+1}(m))\ldots)$ and

$\mid C_{\lambda,t} \mid= \chi_{c_t+1}(\ldots(\chi_{c_1+1}(m+\lambda))\ldots) - \chi_{c_t+1}(\ldots(\chi_{c_1+1}(m+\lambda-1))\ldots)$.

Hence the result holds.□

Now we can state the following interesting corollary.

Corollary 2.4. Let $\mathcal{S}_\ell$ be the variety of solvable groups of length at most ℓ and $G \cong \mathbf{Z}^{(m)} \oplus \mathbf{Z}_{n_1} \oplus \mathbf{Z}_{n_2} \oplus \ldots, \mathbf{Z}_{n_k}$ be a finitely generated abelian group, where $n_{i+1} \mid n_i$ for all $1 \leq i \leq k-1$. Then the following isomorphism holds:

$$\mathcal{S}_\ell M(G) \cong \mathbf{Z}^{(h_m)} \oplus \mathbf{Z}_{n_1}^{(h_{m+1}-h_m)} \oplus \ldots \oplus \mathbf{Z}_{n_k}^{(h_{m+k}-h_{m+k-1})}$$

where $h_i = \chi_2 \underbrace{(\ldots(}_{(l-times)} \chi_2(i))\ldots)$ for all $m \leq i \leq m+k$.

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